## MATH 105 101 Midterm 1 Sample 5

- 1. (15 marks)
  - (a) (4 marks) Given the function:

$$f(x,y) = x\sin y,$$

find  $\frac{\partial^2 f}{\partial x \partial y}$  and  $\frac{\partial^2 f}{\partial y^2}$  at the point  $(2, \pi/6)$ . Simplify your answers.

**Solution:** Calculate the first-order partial derivative of f with respect to y:

$$\frac{\partial f}{\partial y}(x,y) = x\cos y.$$

Then, we get:

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \cos y, \qquad \frac{\partial^2 f}{\partial y^2}(x, y) = -x \sin y.$$

Evaluate those second-order partial derivatives at  $(2, \pi/6)$ :

$$\frac{\partial^2 f}{\partial x \partial y}(2, \pi/6) = \cos \pi/6 = \sqrt{3}/2, \qquad \frac{\partial^2 f}{\partial y^2}(2, \pi/6) = -2\sin \pi/6 = -1.$$

(b) (3 marks) Find all values of a such that the point (a, 1) lies on the level curve of the function  $z = f(x, y) = x^2 + xy$  for  $z_0 = 6$ .

**Solution:** If (a, 1) lies on the level curve of the function  $z = f(x, y) = x^2 + xy$  for  $z_0 = 6$ , then  $6 = f(a, 1) = a^2 + a$ . Solving  $a^2 + a - 6 = 0$ , we get (a-2)(a+3) = 0, so a = 2 or a = -3.

(c) (2 marks) Let  $\mathbf{v} = \langle \frac{3}{4}, -\frac{1}{2} \rangle$ , and  $\mathbf{w} = \langle 16, -6 \rangle$ . Determine if  $\mathbf{v}$  is perpendicular to  $\mathbf{w}$ . Justify your answer.

Solution: We have that:

$$\frac{3}{4}(16) + \left(-\frac{1}{2}\right)(-6) = 12 + 3 = 15 \neq 0,$$

so  ${\bf v}$  is not perpendicular to  ${\bf w}.$ 

(d) (3 marks) A plane  $\mathcal{P}$  is parallel to the plane 2x + z = y and passes through the point P(1, -1, 3). Find the equation of the plane  $\mathcal{P}$ .

**Solution:** A normal vector to the plane 2x + z = y is  $\mathbf{n} = \langle 2, -1, 1 \rangle$ . Since  $\mathcal{P}$  is parallel to 2x + z = y,  $\mathbf{n}$  is also a normal vector of  $\mathcal{P}$ . Thus, an equation of the plane  $\mathcal{P}$  with normal vector  $\mathbf{n}$  going through the point P(1, -1, 3) is:

$$2(x-1) - (y+1) + (z-3) = 0$$
  

$$\Rightarrow 2x - y + z = 6.$$

(e) (3 marks) Given functions:

$$F(x,y) = x + e^y, \qquad G(x,y) = y + e^x,$$

does there exist a function f(x, y) such that  $\nabla f(x, y) = \langle F, G \rangle$ ? Justify your answer and clearly state any result that you may use.

**Solution:** Suppose that f(x, y) exists such that  $\nabla f(x, y) = \langle F, G \rangle$ , that is,  $f_x(x, y) = F(x, y) = x + e^y$ , and  $f_y(x, y) = G(x, y) = y + e^x$ . Then,  $f_{xy}(x, y) = e^y$  and  $f_{yx}(x, y) = e^x$ , which are both continuous on  $\mathbb{R}^2$ . By Clairaut's Theorem,  $f_{xy} = f_{yx}$ . However,  $f_{xy}(x, y) = e^y \neq e^x = f_{yx}(x, y)$ ,

so, no such function f(x, y) exists.

2. (5 marks) Consider the surface S given by:

$$z = 3 + x^2 + 4y^2.$$

(a) (4 marks) Find and sketch the traces of S in the z = 4 and x = 0 planes.

**Solution:** For z = 4, the trace of S is  $4 = 3 + x^2 + 4y^2$ , that is,  $1 = x^2 + 4y^2$ (an ellipse in xy-plane centered at (0,0)).  $y = \frac{x^2}{-h_0 - 0.5} \frac{y}{0.4} \frac{y}{-h_0 - 0.5} \frac{y}{0.5} \frac{x}{-h_0 - 0.5} \frac{y}{-h_0 - 0.5} \frac{x}{-h_0 - 0.5} \frac{x}{-h_$  (b) (1 mark) Based on the traces you sketched above, which of the following renderings represents the graph of the surface?



**Solution:** The answer is (B) since the trace of the surface in (A) for x = 0 looks like two lines, not a parabola. (Alternatively, the surface in (A) contains points with negative values of z, where as the surface given by  $z = 3 + x^2 + 4y^2$  does not contain any such points.)

3. (10 marks) Let R be the triangle whose vertices are (2,0), (-1,0) and (0,2). Find the maximum and minimum values of the function

$$f(x,y) = x^2 - 2x + y.$$

on the boundary of the region R.

**Solution:** The boundary of R consists of three lines. We proceed to find candidates for extrema on each segment:

- On the horizontal line y = 0  $(-1 \le x \le 2)$ : Then,  $g_1(x) = f(x, 0) = x^2 2x$ , and  $g'_1(x) = 2x - 2 = 0$  for x = 1. So, the candidates are (1, 0) (critical point), (-1, 0), (2, 0) (endpoints).
- On the line connecting (2,0) and (0,2): it is given by y = -x+2 for  $0 \le x \le 2$ . So,  $g_2(x) = x^2 - 2x + (-x+2) = x^2 - 3x + 2$ , and  $g'_2(x) = 2x - 3 = 0$  for x = 1.5. So, the candidates are (1.5, 0.5) (critical point), (2,0) and (0,2) (end points).
- On the line connecting (-1,0) and (0,2): it is given by y = 2x + 2 for  $-1 \le x \le 0$ . So,  $g_3(x) = x^2 2x + 2x + 2 = x^2 + 2$ , and  $g'_3(x) = 2x = 0$  for x = 0. So, the candidates are (-1,0) (end point) and (0,2) (end point and critical point).

Evaluate f(x, y) at those points found above, we have:

$$f(1,0) = -1,$$
  $f(-1,0) = 3,$   $f(2,0) = 0,$   $f(1.5,0.5) = -0.25,$   $f(0,2) = 2.$ 

Thus, f(x, y) attains its maximum value of 3 at the point (-1, 0) and its minimum value of -1 at the point (1, 0).

4. (10 marks) Find *all* critical points of the following function:

$$f(x,y) = e^{-\frac{1}{3}x^3 + x - y^2}.$$

Classify each point as a local minimum, local maximum, or saddle point. You do not have to solve for extrema on the boundary.

**Solution:** Compute the first-order partial derivatives of f:

$$f_x(x,y) = e^{-\frac{1}{3}x^3 + x - y^2}(-x^2 + 1) \qquad f_y(x,y) \qquad = e^{-\frac{1}{3}x^3 + x - y^2}(-2y).$$

Since both  $f_x$  and  $f_y$  are defined at every point in  $\mathbb{R}^2$ , the only critical points of f are those at which  $f_x = f_y = 0$ . If  $f_x = 0$ , then  $-x^2 + 1 = 0$  (since the exponential part is always positive), which gives  $x = \pm 1$ . On the other hand,  $f_y = 0$  for y = 0. So, we get two critical points (1, 0) and (-1, 0). Compute the second-order partial derivatives and the discriminants,

$$f_{xx} = (x^4 - 2x^2 - 2x + 1)e^{-\frac{1}{3}x^3 + x - y^2}, \quad f_{yy} = (4y^2 - 2)e^{-\frac{1}{3}x^3 + x - y^2},$$
$$f_{xy} = e^{-\frac{1}{3}x^3 + x - y^2}(-2y)(-x^2 + 1).$$

Using the Second Derivative Test to classify the points, we get:

- At the point (1,0),  $f_{xx}(1,0) = -2e^{2/3} < 0$ ,  $f_{yy}(1,0) = -2e^{2/3}$ ,  $f_{xy}(1,0) = 0$ , so  $D(1,0) = 4e^{4/3} > 0$ , so (1,0) is local maximum.
- At the point (-1,0),  $f_{xx}(-1,0) = 2e^{-2/3}$ ,  $f_{yy}(1,0) = -2e^{-2/3}$ ,  $f_{xy}(1,0) = 0$ , so  $D(1,0) = -4e^{-4/3} < 0$ , so (1,0) is a saddle point.
- 5. (10 marks) A candy company produces boxes of bubblegum and gummy bears. It costs the company \$1 to produce a box of either type of candy. On the other hand, bubblegum sells for \$3 per box and gummy bear for \$5 per box. Due to limitations of sugar supply, the production scheme has to satisfy the production possibilities curve:

$$\sqrt{x} + 2\sqrt{y} = 300,$$

where x and y denote the number of boxes of bubblegum and gummy bears that the company produces weekly. Assuming that the company manages to sell every unit produced, use the method of Lagrange multipliers to answer the following question: how many boxes of each type of candy should the company aim to produce if it is to maximize profit?

Clearly state the objective function and the constraint. You are not required to justify that the solution you obtained is the absolute maximum. A solution that does not use the method of Lagrange multipliers will receive no credit, even if it is correct.

**Solution:** Since it costs the company \$1 to produce a box of either type of candy and bubblegum sells for \$3 per box while gummy bear sells for \$5 per box, we get the following profit function:

$$P(x, y) =$$
Revenue  $-$  Cost  $= (3x + 5y) - (x + y) = 2x + 4y.$ 

The objective function to maximize is the profit function P(x,y) = 2x + 4y, and the constraint function is  $g(x,y) = \sqrt{x} + 2\sqrt{y} - 300 = 0$  (due to sugar supply limitation). Using Lagrange multiplier, we need to solve the following system of equations:

$$\begin{aligned} \nabla P(x,y) &= \lambda \nabla g(x,y) \\ g(x,y) &= 0 \end{aligned}$$

More explicitly, we need to solve:

$$2 = \lambda \frac{1}{2\sqrt{x}}$$
$$4 = \lambda \frac{1}{\sqrt{y}}$$
$$\sqrt{x} + 2\sqrt{y} - 300 = 0.$$

Isolate for  $\lambda$  in the first two equations, we get:

$$\lambda = 4\sqrt{x}, \qquad \lambda = 4\sqrt{y}.$$

Equate the above equations, we get:

$$4\sqrt{x} = 4\sqrt{y} \Rightarrow x = y.$$

Replace x = y in the third equation, we get  $3\sqrt{x} = 300$ , which gives x = 10,000. Thus, x = y = 10,000 and  $\lambda = 400$ . Therefore, to maximize profit while being constrained by  $\sqrt{x} + 2\sqrt{y} = 300$ , the company should produces 10,000 boxes of each type of candies.